Decomposition of any natural number as sum of squares of rational numbers.
https://www.linkedin.com/feed/update/urn:li:activity:6747726162216185857
Prove that for all positive integers n there exist n distinct, positive rational numbers with sum of their squares equal to n.

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Note that for $n = 2$ it is true. Indeed, $\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2 = 2^2$. Consider now for any $n \in \mathbb{N} \setminus \{1\}$ equation $(n-1)^2x^2 + y^2 = n^2$ relatively positive rational x, y.
We have $(n-1)^2x^2 + y^2 = n^2 \Leftrightarrow \dfrac{(n-1)x}{n+y} = \dfrac{n-y}{(n-1)x} \Leftrightarrow$

$((n-1)x, n+y) = t(n-y, (n-1)x), t \in \mathbb{Q}_+ \Leftrightarrow \begin{cases} (n-1)x = t(n-y) \\ n+y = t(n-1)x \end{cases} \Leftrightarrow$

$\begin{cases} (n-1)x + ty = tn \\ t(n-1)x - y = n \end{cases} \Leftrightarrow \begin{cases} x = x(t) := \dfrac{2nt}{(n-1)(t^2+1)} \\ y = y(t) := \dfrac{n(t^2-1)}{t^2+1} \end{cases}$, for any rational

$t > 1$ (to provide $y > 0$).
Note that $x(t) \neq y(t)$ for any rational $t > 1$ because $x(t) = y(t) \Leftrightarrow t^2 - 1 = \dfrac{2t}{n-1} = 0 \Leftrightarrow$
$t - \dfrac{1}{n-1} = \dfrac{\sqrt{n^2-2n+2}}{n-1}$ where $\sqrt{n^2-2n+2}$ is irrational.
For any $n \geq 3$ assuming $r_1^2 + r_2^2 + \ldots + r_{n-1}^2 = (n-1)^2$ for some distinct, positive rational $r_1, r_2, \ldots, r_{n-1}$ we obtain $n^2 = (n-1)^2x^2(t) + y^2(t) =$
$\sum_{k=1}^{n-1} x^2(t)r_k^2 + y^2(t)$. Let t_k be real positive root of equation
$x^2(t)r_k^2 = y^2(t), k = 1, 2, \ldots, n-1$ (quadratic equation $x^2(t)r_k^2 = y^2(t) \Leftrightarrow$
$t^2 - \dfrac{2tr_k}{n-1} - 1 = 0$ always has only one positive real root).
Then for any positive rational $t > 1$ such that $t \notin \{t_1, t_2, \ldots, t_{n-1}\}$ we have $r_kx(t) \neq y(t), k = 1, 2, \ldots, n-1$ and, therefore,
$r_1x(t), r_2x(t), \ldots, r_{n-1}x(t), y(t)$ be distinct, positive rational numbers with sum of their squares equal to n.
Thus, by Math Induction proved stement of the problem.